MATH 448 — Homework 1 — Solutions

1. A fly population has fecundity $f = 0.12$ and death rate $d = 0.05$ per day, with $P_0 = 400$.

(a) By definition, $\Delta P = fP_t - dP_t = (f - d)P_t = 0.07P_t$.

(b) Since $P_{t+1} = P_t + \Delta P = (1 + f - d)P_t$, we get $P_{t+1} = 1.07P_t$, so $\lambda = 1.07$.

(c) An explicit formula is $P_t = \lambda^t P_0 = 400(1.07)^t$. Evaluating at $t = 5$:

$$P_5 = 400(1.07)^5 \approx 561.0.$$

2. Births, deaths, immigration, and emigration contribute bP, $-dP$, iP, and $-eP$ to ΔP respectively, so

$$\Delta P = bP - dP + iP - eP = (b - d + i - e)P.$$

This has the form $\Delta P = rP$ with $r = b - d + i - e$. The population grows exactly when $r > 0$, that is, when $b + i > d + e$: total additions to the population (births plus immigrants) exceed total losses (deaths plus emigrants).

3. A salmon population lays 3000 eggs per female and then all adults die; a fraction s of eggs survive to spawning age.

(a) Every adult dies each generation, so $d = 1$.

(b) Each female is replaced by $3000s$ spawning females, so the effective fecundity is $f = 3000s$. Then

$$\lambda = 1 + f - d = 1 + 3000s - 1 = 3000s,$$

and the model is $P_{t+1} = 3000s\, P_t$.

(c) The population is stable when $\lambda = 1$, that is, $3000s = 1$, so

$$s = \frac{1}{3000} \approx 0.000333.$$

(d) If $s = 0.001$, then $\lambda = 3000(0.001) = 3$. Since $\lambda > 1$, the population grows, tripling every generation.

4. (Allman & Rhodes, 1.1.15 and 1.1.16(a)–(d).) Start from $N_{t+1} = 2N_t$, with one time step equal to one year.

1.1.15. Measuring the population in thousands means dividing both sides by 1000: if $P = N/1000$, then $P_{t+1} = 2P_t$ — the same growth factor as before. A linear model's growth factor does not depend on the units used to count the population, only on the units used to measure time.

1.1.16. Here the population is kept in individuals, but the time step changes.

(a) With a half-year time step, two steps must reproduce one year's growth: if the new growth factor is μ, then $\mu^2 = 2$, so the model is $P_{t+1} = \sqrt{2}\, P_t$.

(b) With a tenth-of-a-year time step, ten steps must reproduce one year's growth: the growth factor is $2^{1/10} \approx 1.072$.

(c) With a step of h years, $1/h$ steps must reproduce one year's growth, so the growth factor is 2^h.

(d) In general, if the original model $N_{t+1} = kN_t$ uses a one-year time step, then the model that agrees with it at whole years but uses an h-year time step is $P_{t+1} = k^h P_t$: after $1/h$ of these steps, the growth factor $\left(k^h\right)^{1/h} = k$ matches one full year.

5. (Table 1.2 data.) An insect population is measured over eleven days:

t	0	1	2	3	4	5	6	7	8	9	10
P_t	.97	1.52	2.31	3.36	4.63	5.94	7.04	7.76	8.13	8.3	8.36

(a) The successive ratios P_{t+1}/P_t, for $t = 0, \dots, 9$, are

$$1.567, \quad 1.520, \quad 1.455, \quad 1.378, \quad 1.283, \quad 1.185, \quad 1.102, \quad 1.048, \quad 1.021, \quad 1.007.$$

(b) No. A model $P_{t+1} = \lambda P_t$ requires these ratios to be (approximately) constant, but instead they fall steadily from about 1.57 toward 1.

(c) The model fits best over the first two or three days, where the ratio stays near 1.5–1.57. By day 8 or 9 the ratio is close to 1, meaning the population has nearly stopped growing.

(d) The per-capita growth rate decreases toward 0 as the population grows. This is density dependence: the population appears to be leveling off toward a carrying capacity, which a constant-λ model cannot capture.

6. Let $r = 0.4$, $K = 500$, and $P_0 = 60$ in the discrete logistic model.

(a) The model is

$$P_{t+1} = P_t\left(1 + 0.4\left(1 - \frac{P_t}{500}\right)\right).$$

(b) Iterating gives $P_1 = 81.12$, $P_2 = 108.304$, and $P_3 = 142.241$.

(c) The per-capita growth rate at $P_0 = 60$ is $0.4(1 - 60/500) = 0.352$, and at $P_2 = 108.304$ it is $0.4(1 - 108.304/500) \approx 0.313$. Since $0.313 < 0.352$, the per-capita growth rate did decrease, as expected: it is a decreasing function of P in this model.

7. The per-capita growth rate is 0.36 at $P = 100$ and 0.12 at $P = 400$.

Since the discrete logistic model assumes $\Delta P/P$ is a linear function of P, the line through these two points has slope

$$\frac{0.12 - 0.36}{400 - 100} = -0.0008,$$

so

$$\frac{\Delta P}{P} = 0.36 - 0.0008(P - 100) = 0.44 - 0.0008P.$$

Comparing with $r(1 - P/K) = r - (r/K)P$ gives $r = 0.44$ and $r/K = 0.0008$, so $K = 0.44/0.0008 = 550$. At $P = 550$, the per-capita growth rate is $0.44 - 0.0008(550) = 0$ — which makes sense, since $P = 550 = K$ is exactly the carrying capacity, where the population neither grows nor declines.

8. Let $r = 1$ and $K = 10$, so $P_{t+1} = F(P_t)$ with $F(P) = P(2 - P/10)$.

(a) $F(P) = 0$ at $P = 0$ and $P = 20$. The vertex occurs where $F'(P) = 2 - P/5 = 0$, i.e. $P = 10$, and $F(10) = 10$, so the vertex is at $(10, 10)$. On $0 \le P \le 20$, F rises from $(0, 0)$ to the vertex $(10, 10)$ and falls symmetrically back to $(20, 0)$.

(b) and (c) Starting from $P_0 = 4$ and cobwebbing:

$$P_1 = F(4) = 6.4, \qquad P_2 = F(6.4) = 8.704, \qquad P_3 = F(8.704) = 9.832,$$

which match the given check values.

9. Let $r = 2.2$, $K = 10$, $P_0 = 5$. Iterating $P_{t+1} = P_t(1 + 2.2(1 - P_t/10))$ gives

$$P_1 = 10.5, \quad P_2 = 9.345, \quad P_3 = 10.692, \quad P_4 = 9.065, \quad P_5 = 10.930.$$

The values do not settle down; instead they oscillate back and forth above and below $K = 10$, and appear to be approaching a repeating 2-cycle rather than a single equilibrium value.

10. Let $r = 1.5$ and $K = 10$.

(a) $P_{t+1} < 0$ exactly when $1 + r(1 - P/K) < 0$, that is, when

$$P > K\Big(1 + \frac{1}{r}\Big) = 10\Big(1 + \frac{2}{3}\Big) = \frac{50}{3} \approx 16.667.$$

(b) With $P_0 = 20$, which exceeds this threshold,

$$P_1 = 20\big(1 + 1.5(1 - 20/10)\big) = 20(1 - 1.5) = -10.$$

(c) Reading $P_1 < 0$ as "the population went extinct" is a convenient interpretation, but it is not a real biological mechanism: nothing in the model describes how or why the population would disappear. A real overcrowded population would instead decline sharply toward a small *positive* number, not go negative. A better model should stay nonnegative for every $P_t \ge 0$ — which is exactly what the Ricker model in §1.4, $F(P) = Pe^{r(1-P/K)}$, does.