MATH 448 — Homework 2 — Solutions

Exercise 3.1. Let $r = 1.2$, $K = 8$. The equilibria of $P_{t+1} = F(P_t) = P_t\big(1 + r(1 - P_t/K)\big)$ are $P^* = 0$ and $P^* = K = 8$. Differentiating,

$$F'(P) = 1 + r - \frac{2rP}{K} = 2.2 - 0.3P.$$

At $P^* = 0$: $F'(0) = 2.2$. Since $|2.2| > 1$, $P^* = 0$ is **unstable**.
At $P^* = 8$: $F'(8) = 2.2 - 0.3(8) = 2.2 - 2.4 = -0.2$. Since $|-0.2| < 1$, $P^* = 8$ is **stable**.

Exercise 3.2.

(a) The single curve in the bifurcation diagram splits into a fork at $r = 2$. This matches the stability calculation: the stretching factor at $P^* = K$ is $1 - r$, and $|1 - r| = 1$ exactly at $r = 2$, which is where $P^* = K$ loses stability and the 2-cycle takes over.

(b) At $r = 2.7$, the diagram shows a dense band of points rather than a small number of distinct values — this is well past the point where the period-doubling cascade completes ($r \approx 2.570$), so the population is in the chaotic regime: it keeps visiting new values rather than settling into a repeating pattern.

Exercise 3.3. With $r = 2.75$, $K = 1$, $P_{t+1} = P_t + 2.75P_t(1 - P_t)$. Iterating and rounding to four decimal places at each step:

t	$P_0 = 0.5$	$P_0 = 0.5001$	difference
0	0.5000	0.5001	0.0001
1	1.1875	1.1876	0.0001
2	0.5752	0.5749	0.0003
3	1.2471	1.2470	0.0001
4	0.3997	0.4000	0.0003
5	1.0595	1.0600	0.0005
6	0.8861	0.8851	0.0010
7	1.1636	1.1648	0.0012
8	0.6401	0.6369	0.0032
9	1.2736	1.2729	0.0007
10	0.3153	0.3176	0.0023
11	0.9090	0.9136	0.0046
12	1.1365	1.1307	0.0058
13	0.7099	0.7243	**0.0144**

(c) The two sequences first differ by more than 0.01 at $t = 13$.

(d) This is not an arithmetic error — it is the expected behavior of a chaotic model. Both sequences are computed from the exact same deterministic rule; the only difference is a starting value 0.0001 apart. For the first several steps the two orbits track closely, but each step amplifies the tiny gap between them a little more, until by $t = 13$ the two orbits have become noticeably different. This amplification of small differences is exactly what *sensitive dependence on initial conditions* means.

Exercise 3.4. For the discrete logistic model, the stretching factor at $P^* = K$ is $1 - r$.

(a) $r = 0.4$: $1 - r = 0.6$. Since $0 < 0.6 < 1$, $P^* = K$ is stable, and since the stretching factor is *positive*, perturbations keep the same sign at every step: **monotone approach**.

(b) $r = 1.6$: $1 - r = -0.6$. Since $|-0.6| < 1$, $P^* = K$ is still stable, but the stretching factor is now *negative*, so the sign of the perturbation flips every step: **damped oscillation**.

(c) $r = 2.2$: $1 - r = -1.2$. Since $|-1.2| > 1$, $P^* = K$ is **unstable**, so the population does not approach K at all; instead it settles into a **stable 2-cycle** (this r is between the 2-cycle and 4-cycle bifurcation points, $2 < r < 2.449$).

Exercise 3.5. False. The model $P_{t+1} = F(P_t)$ is entirely deterministic: the same r, K, and P_0 always produce exactly the same sequence of values, with no randomness anywhere in the rule. What makes chaotic behavior look unpredictable is *sensitive dependence on initial conditions* (Exercise 3.3): any tiny uncertainty in P_0 — even far beyond what a real measurement could avoid — grows over time until it's impossible to say which orbit you're actually on. The unpredictability is practical, not a sign that the process itself is random.