MATH 448 — Homework 3 — Solutions

Exercise 4.1. For the Ricker model, $F(P) = Pe^{r(1-P/K)}$ has equilibria $P^* = 0$ and $P^* = K = 20$ for any r, K. Differentiating,

$$F'(P) = e^{r(1-P/K)}\Big(1 - \frac{rP}{K}\Big), \qquad \text{so} \qquad F'(K) = 1 - r.$$

With $r = 0.5$: $F'(K) = 1 - 0.5 = 0.5$. Since $|0.5| < 1$, $P^* = K = 20$ is **stable**.

Exercise 4.2. With $r = 1$, $A = 2$, $K = 8$,

$$f(P) = P\Big(\frac{P}{2} - 1\Big)\Big(1 - \frac{P}{8}\Big).$$

(a) Setting $f(P) = 0$ gives $P = 0$, $P = A = 2$, and $P = K = 8$.

(b) On each interval, track the sign of the three factors P, $(P/2 - 1)$, $(1 - P/8)$:

- $(0, 2)$: $(+)(-)(+)$, so $f(P) < 0$.
- $(2, 8)$: $(+)(+)(+)$, so $f(P) > 0$.
- $(8, \infty)$: $(+)(+)(-)$, so $f(P) < 0$.

(c) Expanding, $f(P) = -\frac{1}{16}P^3 + \frac{5}{8}P^2 - P$, so $f'(P) = -\frac{3}{16}P^2 + \frac{5}{4}P - 1$. Evaluating,

$$f'(0) = -1, \qquad f'(2) = \tfrac{3}{4}, \qquad f'(8) = -3,$$

matching the phase line in (b): $P = 0$ is **stable**, $P = 2$ is **unstable**, $P = 8$ is **stable**. A population starting below the Allee threshold $A = 2$ is pushed toward $P = 0$ rather than recovering: below the threshold, the population is too sparse to sustain itself and declines to extinction. This is exactly what a strong Allee effect means — a minimum viable population size is required.

Exercise 4.3. False. A system of two coupled continuous variables can oscillate (a limit cycle, as in predator–prey models), but the Poincaré–Bendixson theorem rules out chaos in two dimensions. Continuous-time chaos requires at least three coupled variables.

Exercise 5.3.

(a) Euler's method turns the ODE into the discrete logistic model with r replaced by hr. That model's equilibrium $P^* = K$ is stable exactly when $0 < hr < 2$, so the critical step size is

$$h^* = \frac{2}{r} = \frac{2}{2.5} = 0.8.$$

(b) Running the code: at $h = 0.1$ and $h = 0.5$, P_t approaches $K = 10$ smoothly. At $h = 1$, P_t settles into a persistent oscillation around K rather than converging. At $h = 1.5$, P_t overshoots into negative values and diverges.

(c) Yes. $h = 0.1$ and $h = 0.5$ are both below $h^* = 0.8$ and converge smoothly, while $h = 1$ and $h = 1.5$ are both above it and fail to converge. At $h = 1$, the effective growth parameter is $hr = 2.5$, exactly the value used in the "discrete $r = 2.5$" panel of the Lecture 5 figure — so $h = 1$ is the value that reproduces it.

Exercise 5.4. *Payoff* is the outcome of a single interaction, determined only by the two strategies involved (one entry of the payoff matrix). *Fitness* is the growth rate of a strategy in the population as a whole: it is the average of that strategy's payoffs against all the opponents it actually encounters, weighted by how common each opponent's strategy is. Payoff is a fixed number per pairing; fitness is a function of the population's composition x, built by averaging payoffs over that composition.