MATH 448 — Homework 4 — Solutions

Exercise 5.1. With $V = 4$, $C = 6$, the replicator equation is

$$\frac{dx}{dt} = \frac{1}{2}x(1-x)(4-6x) = 3x^3 - 5x^2 + 2x =: g(x).$$

Differentiating, $g'(x) = 9x^2 - 10x + 2$. At $x^* = 2/3$:

$$g'\Big(\tfrac{2}{3}\Big) = 9\Big(\tfrac{4}{9}\Big) - 10\Big(\tfrac{2}{3}\Big) + 2 = 4 - \tfrac{20}{3} + 2 = -\tfrac{2}{3}.$$

Since $g'(2/3) = -2/3 < 0$, $x^* = 2/3$ is **stable**, confirming the phase-line result from Lecture 5.

Exercise 5.2.

(a) A Stag meets Stag with probability x and Hare with probability $1 - x$, so

$$f_{\text{Stag}}(x) = 4x + 0(1-x) = 4x, \qquad f_{\text{Hare}}(x) = 3x + 3(1-x) = 3.$$

Then $f_{\text{Stag}}(x) - f_{\text{Hare}}(x) = 4x - 3$, and the replicator equation is

$$\frac{dx}{dt} = x(1-x)(4x-3).$$

(b) Equilibria: $x = 0$, $x = 1$, and (from $4x - 3 = 0$) $x = 3/4$. Expanding, $\dot{x} = -4x^3 + 7x^2 - 3x$, so $\frac{d\dot{x}}{dx} = -12x^2 + 14x - 3$. Evaluating:

$$\text{at } x = 0: \ -3, \qquad \text{at } x = 1: \ -1, \qquad \text{at } x = \tfrac{3}{4}: \ \tfrac{3}{4}.$$

So $x = 0$ and $x = 1$ are both **stable**, and $x = 3/4$ is **unstable**.

(c) This is **qualitatively different** from Hawk–Dove. There, both pure populations were unstable and every trajectory converged to the same interior mix, $x^* = V/C$, regardless of where it started. Here, both pure populations (all-Stag, all-Hare) are stable, and the interior point $x = 3/4$ is an unstable threshold separating their basins: which outcome a population reaches depends on whether it starts above or below $x = 3/4$, not on the game alone.

Exercise 6.1. The maximum sustainable constant harvest is

$$H^* = \frac{rK}{4} = \frac{(0.8)(12)}{4} = 2.4.$$

At $H = 2$,

$$P^*_{\pm} = \frac{K}{2}\left(1 \pm \sqrt{1 - \frac{4H}{rK}}\right) = 6\left(1 \pm \sqrt{1 - \frac{8}{9.6}}\right) = 6\left(1 \pm \sqrt{\tfrac{1}{6}}\right),$$

giving $P^*_+ \approx 8.45$ (stable) and $P^*_- \approx 3.55$ (unstable).

Exercise 6.2. The maximum sustainable harvest fraction is $h^* = r = 0.8$. At $h = 0.5$,

$$P^* = K\Big(1 - \frac{h}{r}\Big) = 12\Big(1 - \frac{0.5}{0.8}\Big) = 12(0.375) = 4.5.$$

Exercise 7.1. Running `eig(L)` on the baseline matrix gives dominant eigenvalue $\lambda_1 \approx 0.945$. Since $\lambda_1 < 1$, the population is **declining**, by about 5.5% per year.
Raising the $(3,3)$ entry alone (currently 0.6610) and re-running, the dominant eigenvalue increases slowly at first and crosses 1 once that entry reaches approximately $\mathbf{0.88}$ — an increase of roughly one third over its baseline value. (The historical figure quoted in Lecture 7, a 14% increase, comes from a more careful analysis that raises both the stay-in-stage *and* the advance-to-next-stage probabilities together, from a single underlying survival rate; adjusting the $(3,3)$ entry alone, as this exercise does, is a simplified version of that and needs a larger change to achieve the same effect.)