MATH 448 — Lecture 1

§1.1: Malthusian model of constant population growth

Key facts

A fraction f of the population is added by births each time step and a fraction d is removed by deaths, so

$$\Delta P = fP - dP = (f-d)P, \qquad P_{t+1} = (1+f-d)P_t = \lambda P_t, \qquad P_t = \lambda^t P_0.$$

- $\lambda = 1 + f - d$ is the *finite growth rate*. Since $0 \le d \le 1$ and $f \ge 0$, always $\lambda \ge 0$.
- $\lambda > 1$: growth. $\lambda = 1$: unchanging. $0 \le \lambda < 1$: decay. The dividing line is 1, not 0.
- f may exceed 1 (one insect can lay hundreds of eggs); d cannot exceed 1.
- Constant λ means the ratio P_{t+1}/P_t is the same at every step. That is what to look for in a data table.

(Homework 1, #1–2)

Problem 1.1. Each female of an insect species lays 200 eggs and then all the adults die; of the eggs, 3% survive to become adult females. Measuring t in generations and tracking only females, find the finite growth rate λ and the resulting model $P_{t+1} = \lambda P_t$.

Solution. Every adult dies each generation, so $d = 1$. Each adult female produces $0.03 \times 200 = 6$ adult females, so the effective fecundity is $f = 6$. Hence

$$\lambda = 1 + f - d = 1 + 6 - 1 = 6, \qquad P_{t+1} = 6P_t.$$

Problem 1.2. In the population of Problem 1.1, suppose instead the survival fraction s is unknown, but the population is known to be unchanging ($\lambda = 1$). Find s.

Solution. Since $d = 1$, $\lambda = 1$ forces $f = 1$. With 200 eggs per female, $200s = 1$, so

$$s = 0.005 \quad \text{(one egg in two hundred)}.$$

Problem 1.3. Suppose $N_{t+1} = 2N_t$, where one time step is one year and N is measured in individuals.

(a) Let $P = N/1000$, so the population is measured in thousands. Find the model for P_{t+1} in terms of P_t.

(b) Suppose instead one time step is half a year, with growth factor μ. Find μ, and more generally the growth factor for a step of h years.

Solution.

(a) Dividing both sides of $N_{t+1} = 2N_t$ by 1000 gives $P_{t+1} = 2P_t$: the growth factor is unchanged, since a linear model's growth factor does not depend on the units used to count the population.

(b) Two half-year steps should produce the same effect as one full year, so $\mu^2 = 2$ and $\mu = \sqrt{2} \approx 1.414$; more generally, a step of h years requires growth factor 2^h.

(Homework 1, #3–4)

Problem 1.4. Yeast cells are counted on five successive days:

t	0	1	2	3	4
P_t	12	19	30	47	74

Determine whether these counts are consistent with a model $P_{t+1} = \lambda P_t$ for a single constant λ, and if so, predict the count on day 5.

Solution. Compute the successive ratios P_{t+1}/P_t:

$$1.583, \quad 1.579, \quad 1.567, \quad 1.575.$$

These agree to about 1%, so $P_{t+1} = \lambda P_t$ with $\lambda \approx 1.57$ is a reasonable description here, and the day-5 count should be near $1.57 \times 74 \approx 116$.

If instead the ratios *drift steadily downward*, this is not measurement error: it indicates that the growth factor depends on the population size.

(Homework 1, #5)