MATH 448 — Lecture 2

§1.2: nonlinear models

Key facts

The *per-capita growth rate* is $\Delta P/P$. Assuming it decreases linearly with population size gives the *discrete logistic model*:

$$\frac{\Delta P}{P} = r\left(1 - \frac{P}{K}\right) \qquad \Longleftrightarrow \qquad P_{t+1} = P_t\left(1 + r\left(1 - \frac{P_t}{K}\right)\right).$$

- K is the carrying capacity: $P = K$ gives zero growth; $P < K$ grows, $P > K$ declines.
- When $P \ll K$, r is the (approximate) per-capita growth rate.
- Both ΔP and P_{t+1} are quadratic functions of P_t.
- **Cobwebbing.** Graph $P_{t+1} = F(P_t)$ and the diagonal $P_{t+1} = P_t$. From P_0 on the horizontal axis: go *vertically* to the curve (this evaluates F), then *horizontally* to the diagonal (this copies the answer back to the horizontal axis). Repeat.

Problem 2.1. Suppose the per-capita growth rate $\Delta P/P$ is measured to be 0.30 at $P = 200$ and 0.10 at $P = 600$. Assuming the discrete logistic model, find r and K.

Solution. Since the model assumes $\Delta P/P$ is a linear function of P, the line through these two points has slope $(0.10 - 0.30)/400 = -0.0005$, so

$$\frac{\Delta P}{P} = 0.30 - 0.0005(P - 200) = 0.40 - 0.0005P.$$

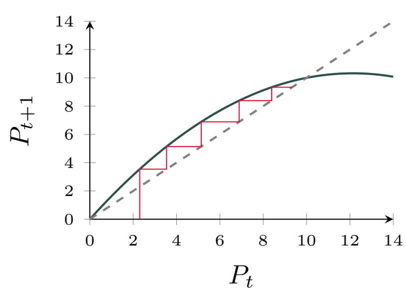
Comparing coefficients with $r(1 - P/K) = r - (r/K)P$ gives

$$r = 0.40, \qquad K = 800.$$

(Homework 1, #6–7)

Problem 2.2. Let $r = 0.7$ and $K = 10$, so that $F(P) = P\big(1 + 0.7(1 - P/10)\big) = 1.7P - 0.07P^2$. Cobweb the model $P_{t+1} = F(P_t)$ starting from $P_0 = 2.3$.

Solution.



t	0	1	2	3	4
P_t	2.30	3.54	5.14	6.89	8.39

(Homework 1, #8–9)

Problem 2.3. Let $F(P) = P\big(1 + r(1 - P/K)\big)$ for constants $r, K > 0$, as in the discrete logistic model. Find every P for which $F(P) < 0$, and evaluate this threshold for $r = 0.7$, $K = 10$ (the values used in Problem 2.2).

Solution. Since the graph of F is a downward-opening parabola, $F(P)$ is negative once P is large enough. Setting $1 + r(1 - P/K) = 0$ and solving for P gives

$$P = K\left(1 + \frac{1}{r}\right),$$

above which $F(P) < 0$; for $r = 0.7$, $K = 10$, this is $P \approx 24.3$.

If P_0 exceeds this threshold, then $P_1 = F(P_0)$ is negative. This does not correspond to any biological mechanism: a real overcrowded population would decline to a small positive number, not a negative one. Section 1.4 replaces F with the Ricker model, $F(P) = Pe^{r(1-P/K)}$, which stays positive for all $P \geq 0$.

(Homework 1, #10 (read §1.4's start first))