MATH 448 — Lecture 3

§1.3: equilibrium, stability, and long-term behavior

Today's question: near an equilibrium, do nearby populations move *toward* it or *away* from it? We explore this by eye first, using the **Cobweb Explorer** (open `CobwebExplorer.html`), then build a test for it.

Exploration 1: the Malthusian model. In Malthusian mode, cobweb a few starting values P_0 with $\lambda = 0.7$: every orbit moves toward 0. Reset and try $\lambda = 1.3$: every orbit (other than $P_0 = 0$ itself) moves away from 0. In both cases $P^* = 0$ is the *only* equilibrium — but whether it attracts or repels nearby populations depends on λ.

Key facts: equilibrium

For a model $P_{t+1} = F(P_t)$, a value P^* is an *equilibrium* if $F(P^*) = P^*$: a population starting exactly at P^* stays there forever. Informally, P^* is *stable* if populations starting near P^* move toward it, and *unstable* if they move away.

Exploration 2: the logistic model. Switch to Logistic mode ($K = 10$) and step through the presets in order: $r = 0.7$, then $r = 1.5$, then $r = 2.3$, then $r = 2.75$. Watch the equilibrium at $P^* = K = 10$ change character each time — this is the progression we'll summarize at the end of today's lecture.

Problem 3.1. In the $r = 0.7$, $K = 10$ model, $P^* = 10$ is an equilibrium (as in Lecture 2, K is where growth is zero), and the exploration above suggests it's stable. Confirm this algebraically: substitute $P_t = 10 + p_t$ for small p_t and find an approximate relation between p_{t+1} and p_t.

Solution. Substituting into $P_{t+1} = P_t\big(1 + 0.7(1 - P_t/10)\big)$,

$$\begin{aligned} P_{t+1} &= (10 + p_t)\Big(1 + 0.7\big(1 - \tfrac{10+p_t}{10}\big)\Big) = (10 + p_t)(1 - 0.07p_t) \\ &= 10 + p_t - 0.7p_t - 0.07p_t^2 = 10 + 0.3p_t - 0.07p_t^2. \end{aligned}$$

So $P_{t+1} = 10 + p_{t+1}$ gives $p_{t+1} = 0.3p_t - 0.07p_t^2 \approx 0.3p_t$ for small p_t: the perturbation shrinks by roughly a factor of 3 each step, matching what the Explorer showed.

Key facts: two stability tests

Write $P_t = P^* + p_t$ for a small perturbation p_t.

- **Noncalculus test.** Substitute into F and drop terms of degree 2 or higher in p_t: this gives $p_{t+1} \approx m\, p_t$ for some constant m, the *stretching factor* (this is what Problem 3.1 just did by hand).
- **Calculus test.** The stretching factor is exactly $F'(P^*)$ — no expansion needed, just one derivative.
- Either way: $|m| < 1 \Rightarrow$ stable; $|m| > 1 \Rightarrow$ unstable; $|m| = 1$ is inconclusive.
- Stability is *local*: it describes P_t close to P^*, not far from it.

Problem 3.2. Redo Problem 3.1 with the calculus test: find $F'(P)$ for the $r = 0.7$, $K = 10$ model and use it to classify *both* equilibria.

Solution. Differentiating $F(P) = P\big(1 + 0.7(1 - P/10)\big)$,

$$F'(P) = 1.7 - 0.14P.$$

So $F'(0) = 1.7$: since $|1.7| > 1$, $P^* = 0$ is unstable. And $F'(10) = 1.7 - 1.4 = 0.3$: since $|0.3| < 1$, $P^* = 10$ is stable, matching Problem 3.1 — but from one derivative instead of an expansion.

Now do Exercise 3.1 from Homework 2.

The general pattern. Problems 3.1 and 3.2 only used $r = 0.7$. The same differentiation works for any r and K: for $F(P) = P\big(1 + r(1 - P/K)\big)$,

$$F'(P) = (1 + r) - \frac{2r}{K}P, \qquad \text{so} \qquad F'(K) = (1 + r) - 2r = 1 - r.$$

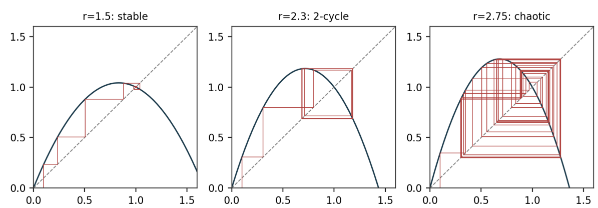
So the stretching factor at $P^* = K$ is always $1 - r$, no matter what K is. This single number is about to explain everything.

Exploration 3: predicting from $F'(K) = 1 - r$. Before touching the Explorer, compute $F'(K) = 1 - r$ at $r = 1.5$ and at $r = 2.3$, and for each decide: is $P^* = K$ stable or unstable, and if stable, should the approach be monotone or oscillating? Write your predictions down. Then reopen the Explorer (Logistic mode, $K = 10$) and check both against the presets $r = 1.5$ and $r = 2.3$.

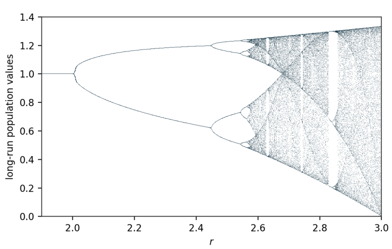
Exploration 4: beyond stability. Continue past what you just predicted — try $r = 2.6$, then $r = 2.75$, and just watch. No computation needed yet; the pattern is explained in the Summary below. (The **Bifurcation Explorer** at `badaljoshi.net/bifurcationexplorer` shows the cobweb, the time series, and the bifurcation diagram together in one view — click or drag anywhere on the diagram to set r.)

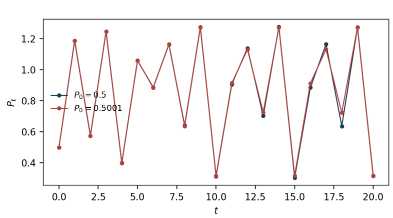
Summary: oscillation, bifurcation, and chaos

- $0 < r \leq 1$: monotone approach to $P^* = K$.
- $1 < r < 2$: *damped oscillation* toward $P^* = K$ (the stretching factor $1-r$ is negative, so overshoots alternate sides).
- $r = 2$: $|1 - r| = 1$; $P^* = K$ is about to lose stability.
- $r > 2$: $P^* = K$ is unstable; the population settles into a stable *2-cycle* instead.
- Further increases: the 2-cycle loses stability to a 4-cycle, then an 8-cycle, and so on (*period doubling*).
- $r \gtrsim 2.570$ (in the units where $K = 1$): *chaotic* for most r — deterministic, but so sensitive to P_0 that long-term prediction is impossible in practice.



The three regimes from Explorations 3–4, frozen at $r = 1.5$, 2.3, and 2.75 (here with $K = 1$): stable with damped oscillation, a 2-cycle, and chaos.





The bifurcation diagram (top): for each r, many iterations are run, the first few hundred (the *transient*) are discarded, and the remaining values of P_t are plotted above that r. A single curve means a stable equilibrium; a fork means a stable cycle; a dense band means chaos.

Sensitivity to initial conditions at $r = 2.75$ (bottom): two orbits starting just 0.0001 apart track together at first, then diverge completely by $t \approx 15$. This — not randomness — is what makes the behavior chaotic. The Cobweb Explorer's *Compare a second orbit* mode reproduces this live.

Now do Exercises 3.2–3.5 from Homework 2.