MATH 448 — Lecture 5

Euler's method; an introduction to evolutionary game dynamics

Today: connect last time's continuous model back to the discrete one, then apply the same equilibrium/stability toolkit to a different kind of question — not population size, but which strategy prevails.

Euler's method

To approximate a differential equation on a computer, replace $\frac{dP}{dt}$ by the difference quotient $\frac{\Delta P}{h}$ over a small time step h:

$$\frac{P_{t+1} - P_t}{h} \approx f(P_t) \quad \Longrightarrow \quad P_{t+1} = P_t + h\, f(P_t).$$

This is *Euler's method*: it turns any ODE $dP/dt = f(P)$ into a discrete model.

Problem 5.1. Apply Euler's method with step size h to the logistic ODE $dP/dt = rP(1 - P/K)$.

Solution.

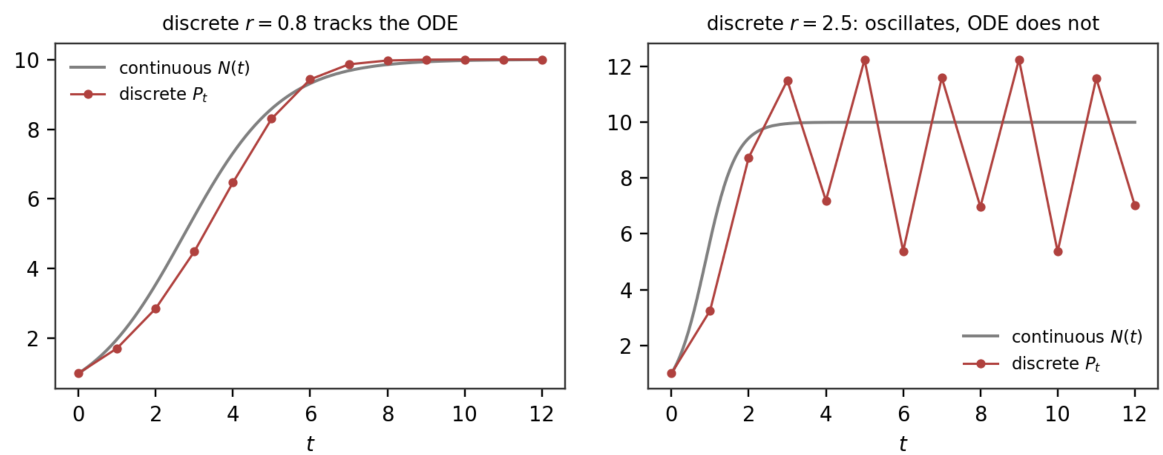
$$P_{t+1} = P_t + h \cdot rP_t\Big(1 - \frac{P_t}{K}\Big) = P_t\Big(1 + hr\Big(1 - \frac{P_t}{K}\Big)\Big).$$

Problem 5.2. What happens to Problem 5.1's formula when $h = 1$?

Solution. Setting $h = 1$,

$$P_{t+1} = P_t\Big(1 + r\Big(1 - \frac{P_t}{K}\Big)\Big),$$

which is exactly the discrete logistic model from Lecture 2: the discrete model used throughout the course is Euler's method applied to the continuous logistic ODE with step size $h = 1$.



Both panels use the same P_0, r, and K. Left: for small r, the $h = 1$ discrete steps closely track the continuous solution. Right: for larger r, the continuous solution still approaches K smoothly (as Lecture 4 established for every r), but the discrete steps overshoot and oscillate, since the step size delays the population's response to overcrowding by a full step. Repeated overshoot of this kind produces the bifurcations and chaos of Lecture 3.

Now do Exercise 5.3 from Homework 3.

A population with variation. Every model so far has assumed a uniform population: all individuals share the same growth rate. We now allow variation, considering different behaviors or types within a population, which we call *strategies*.

Terms

- **Strategy**: a distinct behavior or type present in the population.
- **Frequency** x: the fraction of the population using a given strategy (so $1 - x$ uses the other, in a two-strategy population).
- **Fitness** $f(x)$: the growth rate of individuals using a given strategy — the same idea as the per-capita growth rate r from earlier, but now allowed to depend on the population's composition x.
- **Payoff**: the contribution to fitness from a single interaction, determined by the strategies of both individuals involved.

Now do Exercise 5.4 from Homework 3.

Deriving the replicator equation. Suppose the population consists of two types, A and B, with sub-population sizes $N_A(t)$ and $N_B(t)$, each growing at its own fitness:

$$\dot{N}_A = f_A(x)N_A, \qquad \dot{N}_B = f_B(x)N_B,$$

where $N = N_A + N_B$ and $x = N_A/N$ is the frequency of type A. By the quotient rule,

$$\dot{x} = \frac{\dot{N}_A N - N_A \dot{N}}{N^2} = \frac{f_A N_A N - N_A(f_A N_A + f_B N_B)}{N^2} = \frac{N_A N_B (f_A - f_B)}{N^2} = x(1-x)\big(f_A(x) - f_B(x)\big).$$

The replicator equation

$$\frac{dx}{dt} = x(1-x)\big(f_A(x) - f_B(x)\big).$$

$x = 0$ and $x = 1$ are always equilibria (only one strategy present); there may also be interior equilibria where $f_A(x) = f_B(x)$.

Hawk–Dove. Two animals meet over a resource worth V. Each plays *Hawk* (escalate, fight to keep it) or *Dove* (display, then retreat if the opponent escalates).

- **Dove vs. Dove.** Neither escalates; one of them eventually gets the resource, say with probability $\frac{1}{2}$ each, no injury. Expected payoff: $V/2$.
- **Hawk vs. Dove.** Hawk escalates, Dove retreats immediately — no fight. Hawk gets V; Dove gets 0.
- **Hawk vs. Hawk.** Both escalate, so they fight. Each wins with probability $\frac{1}{2}$; the winner gets V, the loser pays an injury cost C. Expected payoff: $(V - C)/2$.

	Hawk	Dove
Hawk	$(V-C)/2$	V
Dove	0	$V/2$

Assume $C > V$: losing a fight costs more than the resource is worth, the relevant case whenever fighting carries real risk.

Problem 5.3. Let x be the frequency of Hawk. Using the payoff matrix above, find $f_H(x) - f_D(x)$, write down the replicator equation, find all equilibria, and (assuming $C > V$) determine the stability of each using the phase line.

Solution. A Hawk meets Hawk with probability x and Dove with probability $1 - x$, so

$$f_H(x) = x \cdot \frac{V - C}{2} + (1-x)V, \qquad f_D(x) = x \cdot 0 + (1-x)\frac{V}{2}.$$

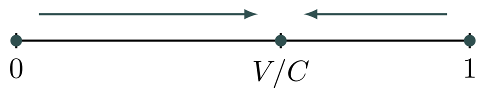
Subtracting,

$$f_H(x) - f_D(x) = \frac{V - Cx}{2},$$

so the replicator equation is

$$\frac{dx}{dt} = \frac{1}{2}\, x(1-x)(V - Cx).$$

Equilibria: $x = 0$, $x = 1$, and (from $V - Cx = 0$) $x^* = V/C$, which lies in $(0, 1)$ precisely when $C > V$.
For the phase line: $x(1 - x) > 0$ throughout $(0, 1)$, so the sign of dx/dt matches the sign of $V - Cx$, positive for $x < V/C$ and negative for $x > V/C$; trajectories on both sides move toward $x^* = V/C$, so it is stable. Near $x = 0$, $dx/dt \approx xV/2 > 0$: unstable. Near $x = 1$, since $C > V$, $dx/dt < 0$ just below 1: also unstable. Every trajectory in $(0, 1)$ therefore converges to $x^* = V/C$.



Phase line for Hawk–Dove ($C > V$): $x = 0$ and $x = 1$ are unstable; every trajectory in $(0, 1)$ converges to $x^* = V/C$.

A concrete case. Let $V = 4$ and $C = 6$ (so $C > V$). Then

$$x^* = \frac{V}{C} = \frac{4}{6} = \frac{2}{3} :$$

two-thirds of the population plays Hawk at equilibrium. In general, the evolutionarily stable fraction of aggressive individuals equals the value of the resource divided by the cost of injury, $x^* = V/C$.

Now do Exercise 5.1 from Homework 3.

Now do Exercise 5.2 from Homework 3.

Across three lectures, the biology has changed — discrete population growth, continuous population growth, continuous frequency dynamics — but the equilibrium-and-stability method has not.