MATH 448 — Lecture 6

Harvesting and bifurcation in continuous models

Today opens by finishing Hawk–Dove from Lecture 5 (Problem 5.3 onward; see that handout). Once that is resolved: a new kind of bifurcation, arising this time in a continuous model.

Scenario. A population grows logistically, but is also harvested — fished, hunted, or otherwise removed. In small groups, propose how the logistic ODE should be modified to describe: (a) removing a *fixed number* of individuals per unit time, and (b) removing a *fixed fraction* of the population per unit time.

Constant-yield harvesting

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - H, \qquad H > 0.$$

H is a fixed number of individuals removed per unit time, independent of P.

Problem 6.1. Find the equilibria of the constant-harvest model in terms of r, K, H, and classify each using the phase line, assuming two distinct equilibria exist.

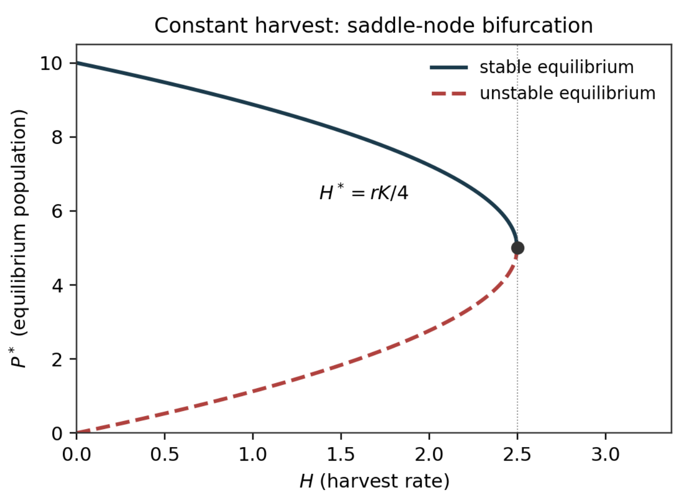
Solution. Equilibria satisfy $rP(1 - P/K) = H$, i.e. $\frac{r}{K}P^2 - rP + H = 0$. By the quadratic formula,

$$P^*_\pm = \frac{K}{2}\left(1 \pm \sqrt{1 - \frac{4H}{rK}}\right),$$

real and distinct provided $H < rK/4$. Writing $f(P) = -\frac{r}{K}(P - P^*_-)(P - P^*_+)$: since the leading coefficient is negative, $f(P) < 0$ for $P < P^*_-$, $f(P) > 0$ for $P^*_- < P < P^*_+$, and $f(P) < 0$ for $P > P^*_+$. So P^*_+ is **stable** and P^*_- is **unstable**: a population starting below P^*_- declines to extinction, while one starting above it settles at P^*_+.

Problem 6.2. Show that the two equilibria in Problem 6.1 collide when $H = rK/4$, and describe what happens for H larger than this value.

Solution. The discriminant of $\frac{r}{K}P^2 - rP + H = 0$ is $r^2 - \frac{4rH}{K}$, which vanishes exactly when $H = \frac{rK}{4}$. Beyond this value the discriminant is negative: no real equilibria exist, so $f(P)$ has no root and (since $f(P) \to -\infty$ as $P \to \infty$) must be negative for every $P \geq 0$. The population declines to extinction regardless of its starting size — there is no threshold to stay above, because there is no longer anywhere to settle.



Equilibria of the constant-harvest model as functions of H (here $r = 1$, $K = 10$). The two branches meet and vanish at $H^* = rK/4$: a saddle-node *(or* fold*) bifurcation.*

Now do Exercise 6.1 from Homework 4.

Proportional harvesting.

Proportional harvesting

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - hP = P\left[(r-h) - \frac{r}{K}P\right], \qquad h > 0.$$

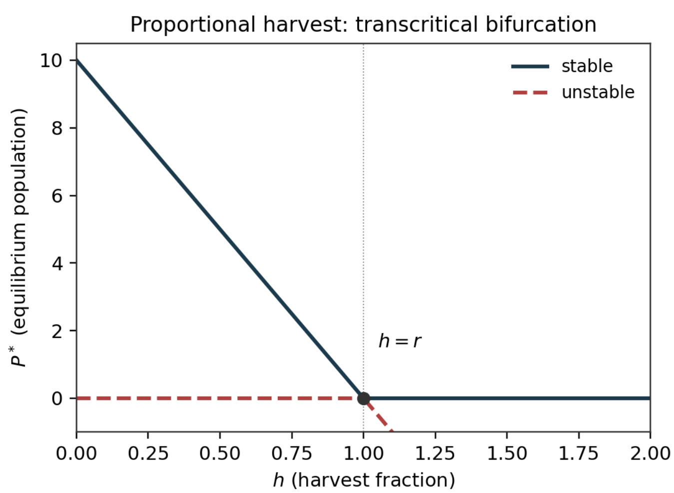
h is the fraction of the population removed per unit time.

Problem 6.3. Find the equilibria of the proportional-harvest model, find $f'(P)$ at each, and describe what happens as h increases past r.

Solution. Equilibria: $P = 0$ and $P^* = K(1 - h/r)$, the latter positive only when $h < r$. Differentiating $f(P) = (r-h)P - \frac{r}{K}P^2$,

$$f'(P) = (r-h) - \frac{2r}{K}P, \qquad \text{so} \qquad f'(0) = r - h, \qquad f'(P^*) = h - r = -(r-h).$$

The two values are always negatives of each other. For $h < r$: $P = 0$ is unstable and $P^* > 0$ is stable, so the population settles at a reduced but positive level. For $h > r$: $P^* < 0$ is no longer meaningful, and $P = 0$ has become stable — the population declines to extinction. At $h = r$ the two equilibria coincide at $P = 0$ and $f'(0) = 0$: the derivative test is inconclusive exactly at the bifurcation point, as it must be.



Equilibria of the proportional-harvest model as functions of h (here $r = 1$, $K = 10$). The two branches cross at $h = r$ and exchange stability rather than vanishing: a transcritical *bifurcation.*

Now do Exercise 6.2 from Homework 4.

Comparison. The two policies fail in different ways. Constant-yield harvesting is safe up to a hard ceiling, $H^* = rK/4$, and then fails catastrophically: crossing the threshold by any amount drives the population to extinction with no warning from its current size. Proportional harvesting fails gracefully: the equilibrium population shrinks continuously as $h \to r$, and there is no hidden threshold below which the population is already doomed. A fixed quota (constant harvest) and a fixed percentage (proportional harvest) are not just two numbers on the same policy — they are two different kinds of risk.