MATH 448 — Lecture 7

Structured populations: projection matrices and eigenvalues

Yesterday: one variable, and bifurcation as a single parameter varies. Today: several variables at once — a population broken into life stages — and a new mathematical tool built for exactly that: matrices and eigenvalues.

Scenario. A bird has two life stages: *juveniles*, who cannot yet breed, and *adults*, who breed every year. A juvenile is a juvenile for exactly one year: at the end of that year it either survives and becomes an adult, or it dies. Adults have their own survival rate and remain adults for the rest of their lives; each adult produces some average number of offspring per year. In small groups, propose a model tracking both the number of juveniles and the number of adults from one year to the next.

Terms

- **Stage vector** n_t: a column vector listing the abundance of each stage at time t.
- **Projection matrix** L: the matrix such that $n_{t+1} = Ln_t$, built from each stage's survival and reproduction.
- **Dominant eigenvalue** λ_1: the long-term growth rate of the whole population, once transients die out.
- **Stable stage distribution**: the eigenvector corresponding to λ_1 — the proportions each stage settles into, regardless of the starting distribution n_0.

Aside: why the dominant eigenvalue is the growth rate

Write the starting vector as a combination of L's eigenvectors, $n_0 = c_1v_1 + c_2v_2 + \cdots$, with v_1 the eigenvector of the dominant eigenvalue λ_1. Since $Lv_i = \lambda_i v_i$, applying L repeatedly gives

$$n_t = L^t n_0 = c_1\lambda_1^t v_1 + c_2\lambda_2^t v_2 + \cdots = \lambda_1^t \left(c_1v_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^t v_2 + \cdots \right).$$

If λ_1 is strictly larger in magnitude than every other eigenvalue, each ratio $(\lambda_i/\lambda_1)^t \to 0$ as $t \to \infty$. So for large t, $n_t \approx c_1\lambda_1^t v_1$: the population's total size scales like λ_1^t each year, and its shape converges to a multiple of v_1 — exactly the growth rate and stable stage distribution claimed above.

Problem 7.1. Using the class's proposed equations for J_{t+1} and A_{t+1}, write the model as a single matrix equation $n_{t+1} = Ln_t$ with $n_t = \begin{pmatrix} J_t \\ A_t \end{pmatrix}$.

Solution. With fecundity f (offspring per adult), juvenile-to-adult survival s_J, and adult survival s_A, the natural proposal is $J_{t+1} = fA_t$ and $A_{t+1} = s_JJ_t + s_AA_t$. In matrix form,

$$\begin{pmatrix} J_{t+1} \\ A_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & f \\ s_J & s_A \end{pmatrix} \begin{pmatrix} J_t \\ A_t \end{pmatrix}.$$

Problem 7.2. Let $f = 2$, $s_J = 0.3$, $s_A = 0.5$. Find both eigenvalues of L by hand, and interpret the dominant one and its eigenvector.

Solution. The characteristic equation of L is $\lambda^2 - s_A\lambda - fs_J = 0$, i.e. $\lambda^2 - 0.5\lambda - 0.6 = 0$. By the quadratic formula,

$$\lambda = \frac{0.5 \pm \sqrt{0.25 + 2.4}}{2} = \frac{1}{4} \pm \frac{\sqrt{265}}{20} \approx 1.064 \text{ or } -0.564.$$

The dominant eigenvalue is $\lambda_1 \approx 1.064$: the population grows about 6.4% per year once it settles down. Its eigenvector solves $(L - \lambda_1 I)v = 0$, giving $J : A \approx 1.88 : 1$ — the stable ratio of juveniles to adults, regardless of how the population started.

Loggerhead sea turtles: a real-world application. The toy model above has two stages; real applications often need many more, and hand computation stops being an option. In 1987, Crouse, Crowder, and Caswell built a seven-stage model for the loggerhead sea turtle (*Caretta caretta*), then listed as threatened.[1] Conservation effort at the time focused almost entirely on protecting eggs on nesting beaches. Their model asked whether that focus made sense.

The seven stages are eggs/hatchlings, small juveniles, large juveniles, subadults, and three adult classes, with projection matrix (their Table 4):

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 127 & 4 & 80 \\ 0.6747 & 0.7370 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0486 & 0.6610 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0147 & 0.6907 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0518 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8091 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8091 & 0.8089 \end{pmatrix}.$$

```
L = [0 0 0 0 127 4 80 ;
     0.6747 0.7370 0 0 0 0 0 ;
     0 0.0486 0.6610 0 0 0 0 ;
     0 0 0.0147 0.6907 0 0 0 ;
     0 0 0 0.0518 0 0 0 ;
     0 0 0 0 0.8091 0 0 ;
     0 0 0 0 0 0.8091 0.8089];
eigenvalues = eig(L)
[V, D] = eig(L);
```

Now do Exercise 7.1 from Homework 4.

The real answer: the baseline model gives $\lambda_1 = 0.945$ (declining about 5.7% per year). Raising egg survival all the way to 100% barely helps. But increasing survival of *large juveniles* by just 14% (from 0.661 to 0.77) is enough to stabilize the population. This is why Turtle Excluder Devices — which reduce juvenile and adult drowning in shrimp trawls, not egg mortality — became the recommended intervention.[2]

One more direction. A game with three strategies instead of two needs a vector of frequencies, and checking the stability of an interior equilibrium means finding the eigenvalues of a Jacobian matrix — the same tool introduced today, applied back to Hawk–Dove-style dynamics.*

*Potential project topic!

[1] D. T. Crouse, L. B. Crowder, and H. Caswell, "A Stage-Based Population Model for Loggerhead Sea Turtles and Implications for Conservation," *Ecology* **68**(5), 1987, pp. 1412–1423.

[2] The original sensitivity analysis recomputed both the stay-in-stage and advance-to-next-stage probabilities from an underlying survival rate for each stage; adjusting a single matrix entry directly, as in Exercise 7.1, is a simplified version of the same idea.