MATH 448 — Lecture 8

Linear systems in the plane: Romeo and Juliet

So far, every continuous model has had one variable. Today: two variables at once, coupled together, in the simplest possible way — linearly. The tool is the same one introduced for structured populations, eigenvalues, aimed at a new kind of system.

Terms

- **Trace** T and **determinant** D of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $T = a + d$, $D = ad - bc$.
- **Node**: both eigenvalues real and the same sign. **Saddle**: both eigenvalues real, opposite signs. **Spiral**: eigenvalues complex with nonzero real part. **Center**: eigenvalues purely imaginary.

General linear systems in the plane

For $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$, the origin is the only equilibrium whenever $\det A \neq 0$. Expanding $\det(A - \lambda I) = 0$ gives the characteristic equation

$$\lambda^2 - T\lambda + D = 0, \qquad T = \operatorname{tr} A, \quad D = \det A.$$

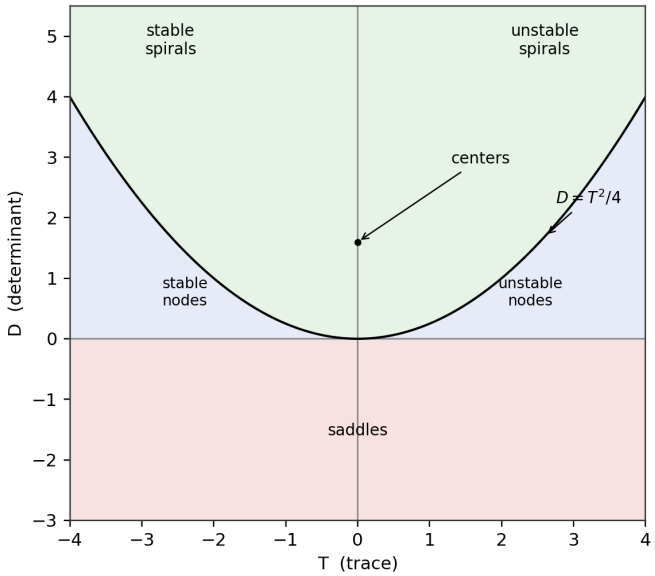
By the quadratic formula, the two eigenvalues sum to T and multiply to D (Vieta's formulas). Classify in three steps:

(a) **Sign of D.** $D < 0$: the eigenvalues are real with opposite signs (their product is negative) — a **saddle**, always unstable, regardless of T. $D > 0$: both eigenvalues share a sign, or are complex conjugates — proceed to (b). ($D = 0$: a degenerate borderline case, at least one eigenvalue is exactly zero; not discussed further here.)

(b) **Sign of T** (when $D > 0$). $T < 0$: **stable**. $T > 0$: **unstable**. $T = 0$: a **center** — purely imaginary eigenvalues, perpetual oscillation, neither stable nor unstable.

(c) **Real or complex** (when $T \neq 0$). $T^2 - 4D \geq 0$: real eigenvalues — a **node**. $T^2 - 4D < 0$: complex eigenvalues — a **spiral**.

So, for instance, "$T < 0$ and $T^2 - 4D < 0$" is a stable spiral: first stability from T, then shape from the discriminant.



The trace–determinant plane. Every 2×2 linear system lands in exactly one region.

Romeo and Juliet.[1] Let $R(t)$ be Romeo's love for Juliet and $J(t)$ be Juliet's love for Romeo (negative values mean dislike). A general linear model of their relationship is

$$\frac{dR}{dt} = aR + bJ, \qquad \frac{dJ}{dt} = cR + dJ,$$

where each constant describes how one person's feelings respond to — their own feelings (a, d) or the other person's (b, c).

Romantic styles

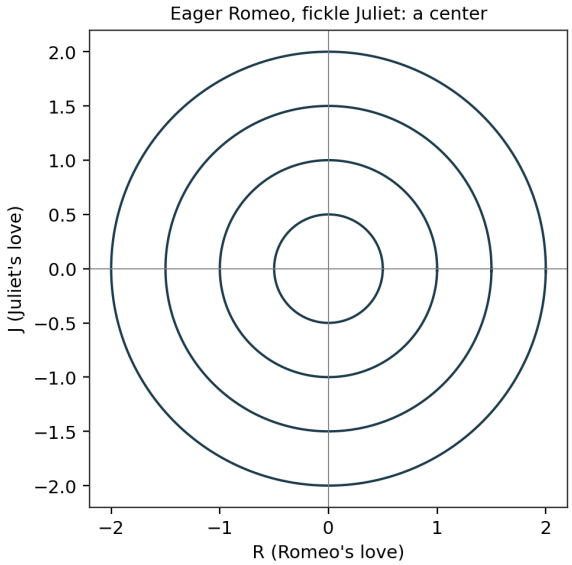
Strogatz names four styles by the signs of a person's own-feeling term and their partner's-feeling term:

- **Eager beaver** (both positive): spurred on by their own feelings and by the other's.
- **Narcissistic nerd** (own positive, partner's negative): wants more of what they already feel, but retreats from the other's advances.
- **Cautious** (or **secure**) **lover** (own negative, partner's positive): retreats from their own feelings, but is drawn in by the other's.
- **Hermit** (both negative): retreats from both.

Problem 8.1 below sits at $a = 0$, a boundary case between styles. Problem 8.2 pairs two identical cautious lovers. Pairing two *different* styles — an eager beaver Romeo with a hermit Juliet, say — gives many more scenarios to work out.

Problem 8.1. *Eager Romeo, fickle Juliet.* Romeo's feelings simply mirror Juliet's ($dR/dt = bJ$, $b > 0$); Juliet, however, backs away the more Romeo loves her ($dJ/dt = cR$, $c < 0$). Find the eigenvalues of the resulting system, classify the equilibrium, and describe the relationship.

Solution. Here $a = d = 0$, so $T = 0$ and $D = -bc$. Since $b > 0$ and $c < 0$, $D = -bc > 0$, and $T^2 - 4D = -4D < 0$: complex eigenvalues with real part $T/2 = 0$ — a **center**. Neither Romeo's nor Juliet's feelings ever settle down or run away: (R, J) traces a closed loop forever, cycling endlessly between mutual love and mutual hate.



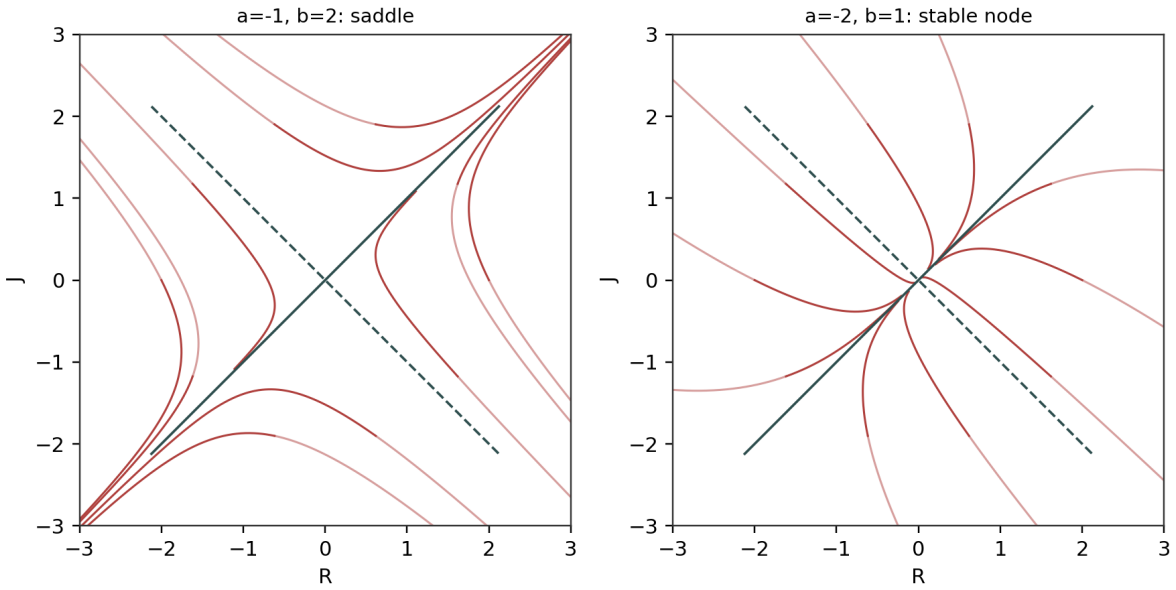
Closed orbits for $b = c = 1$ (taking $c = -1$ in the sign convention above): perpetual cycling, never approaching or leaving the origin.

Cautious lovers. Now suppose both people are identically cautious about their own feelings ($a = d < 0$) but respond equally to each other ($b = c > 0$): $\dot{R} = aR + bJ$, $\dot{J} = bR + aJ$.

[1] S. H. Strogatz, "Love affairs and differential equations," *Mathematics Magazine* **61**(1), 1988, p. 35; also *Nonlinear Dynamics and Chaos*, Addison-Wesley, 1994, §5.3.

Problem 8.2. Find the eigenvalues of this system in terms of a and b, and determine the condition on a, b under which the relationship is a saddle rather than a stable node.

Solution. The matrix $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ is symmetric, and two of its eigenvectors can be found by inspection: for $v = (1, 1)$, $Av = (a + b)v$; for $v = (1, -1)$, $Av = (a - b)v$. So the eigenvalues are exactly $a + b$ and $a - b$ — no characteristic equation needed. Since $a < 0$, both would be negative if $b = 0$; increasing b pushes $a + b$ up and $a - b$ down. In terms of T, D: $T = 2a < 0$, and $D = a^2 - b^2$. The relationship is a **saddle** when $D < 0$, i.e. when $b > |a|$: responsiveness outweighs caution, and whether the couple ends up in mutual love or mutual hate depends sensitively on how they feel at the very start. It is a **stable node** when $b < |a|$: caution wins, and any starting feelings decay to mutual indifference.



Left: $a = -1, b = 2$ ($b > |a|$), a saddle. Right: $a = -2, b = 1$ ($b < |a|$), a stable node. Solid and dashed lines mark the eigenvector directions $R = J$ and $R = -J$.

MATLAB: phase portraits

```
A = [0 1; -1 0]; % change this matrix to try other cases
[R,J] = meshgrid(-3:0.4:3, -3:0.4:3);
dR = A(1,1)*R + A(1,2)*J;
dJ = A(2,1)*R + A(2,2)*J;
quiver(R,J,dR,dJ)
hold on
for R0 = [-2 -1 1 2]
    [t,Y] = ode45(@(t,Y) A*Y, [0 10], [R0;0]);
    plot(Y(:,1), Y(:,2))
end
xlabel('R'); ylabel('J')
```

Now do Exercise 8.1 from Homework 5.

Now do Exercise 8.2 from Homework 5.

What if there were three people in the love affair instead of two?*

*Potential project topic! With a third person and nonlinear terms, love triangles can become chaotic — consistent with Lecture 4's claim that chaos needs at least three coupled continuous variables.