MATH 448 — Practice Midterm 1

Closed book, closed notes. Calculators allowed. 75 minutes. 50 points.

Name: _____

Write your work and final answers directly in the space provided after each part. The score box beside each part shows its point value. If you need more room, use the extra page at the end and write the problem and part number you are continuing.

Reference: equilibria and stability

Every stability question follows the same four steps:

(i) Find all equilibria.

(ii) Linearize near each equilibrium (find the Jacobian there).

(iii) Find the eigenvalues of the linearization.

(iv) Decide stability from the eigenvalues.

Only step (iv) depends on the type of model.

- **Continuous time:** stable if all eigenvalues have negative real part, unstable if any has positive real part.
- **Discrete time:** stable if all eigenvalues have absolute value less than 1, unstable if any has absolute value greater than 1.

Model	**Equilibria**	**Linearization**	**Eigenvalue(s)**	**Stable if**	**Unstable if**
Discrete, one variable: $P_{t+1} = F(P_t)$	Solve $F(P) = P$	$p_{t+1} \approx F'(P^*)\, p_t$	$F'(P^*)$	$\lvert F'(P^*)\rvert < 1$	$\lvert F'(P^*)\rvert > 1$
Continuous, one variable: $dP/dt = f(P)$	Solve $f(P) = 0$	$dp/dt \approx f'(P^*)\, p$	$f'(P^*)$	$f'(P^*) < 0$	$f'(P^*) > 0$
Discrete, linear: $n_{t+1} = Ln_t$	$n = 0$	Already linear	Eigenvalues of L ($\lambda^2 - T\lambda + D = 0$)	Every $\lvert\lambda\rvert < 1$	Some $\lvert\lambda\rvert > 1$
Continuous, two variables: $dx/dt = f(x, y)$, $dy/dt = g(x, y)$	Solve $f = 0$ and $g = 0$ together	Jacobian J at the equilibrium	Eigenvalues of J ($\lambda^2 - T\lambda + D = 0$)	Every eigenvalue has negative real part	Some eigenvalue has positive real part

Problem 1 Score: _____ / 10

A population follows

$$P_{t+1} = P_t\Big(1 + r\big(1 - P_t/K\big)\Big), \qquad r = 1.4,\ K = 15.$$

(a) Find both equilibria. ____ / 3

(b) Classify the stability of each using $|F'(P^*)|$. ____ / 7

Problem 2 Score: _____ / 11

A population subject to a strong Allee effect follows

$$\frac{dP}{dt} = rP\Big(\frac{P}{A} - 1\Big)\Big(1 - \frac{P}{K}\Big), \qquad r = 1,\ A = 3,\ K = 10.$$

(a) Find all three equilibria. ____ / 4

(b) Using the phase line, classify the stability of each. ____ / 5

(c) In one sentence, explain what happens to a population that starts below $P = 3$. ____ / 2

Problem 3 Score: _____ / 10

A population has two stages, juveniles and adults. Each year, 20% of juveniles survive to become adults; 50% of adults survive and remain adults; each adult produces, on average, 2.5 new juveniles.

(a) Write the projection matrix L for $n_{t+1} = Ln_t$, $n_t = \binom{J_t}{A_t}$. ____ / 4

(b) Find both eigenvalues of L. ____ / 5

(c) Is this population growing, shrinking, or exactly stable in the long run? ____ / 1

Problem 4 Score: _____ / 9

A linear system is given by

$$\frac{dR}{dt} = -R + 4J, \qquad \frac{dJ}{dt} = -2R - 3J.$$

(a) Find $T = \operatorname{tr} A$ and $D = \det A$ for this system. ____ / 4

(b) Classify the equilibrium at the origin (node, saddle, spiral, or center; stable or unstable where relevant). Show the steps you used, not just the final answer. ____ / 5

Problem 5 Score: _____ / 10

A predator–prey system is given by

$$\frac{dx}{dt} = x(\alpha - \beta y), \qquad \frac{dy}{dt} = y(\delta x - \gamma), \qquad \alpha = 3,\ \beta = 1,\ \gamma = 2,\ \delta = 0.5.$$

(a) Find the coexistence equilibrium (x^*, y^*). ____ / 3

(b) Find the Jacobian at this equilibrium, and find T and D. ____ / 5

(c) Classify the equilibrium. ____ / 2

Extra space

Write the problem and part number you are continuing.