MATH 448 — Practice Midterm 1 — Solutions

Problem 1 — 10 points

$P_{t+1} = F(P_t)$ *with* $F(P) = P\big(1 + r(1 - P/K)\big)$, $r = 1.4$, $K = 15$.

(a) **[3 pts]**

An equilibrium is a population that does not change from one step to the next, so $F(P^*) = P^*$:

$$P\big(1 + r(1 - P/K)\big) = P \quad \Longrightarrow \quad rP\big(1 - P/K\big) = 0.$$

A product is zero only when one of its factors is zero, so $P = 0$ or $1 - P/K = 0$. The equilibria are

$$P^* = 0 \quad \text{and} \quad P^* = K = 15.$$

(b) **[7 pts]**

Expand $F(P) = P + rP - rP^2/K$ and differentiate:

$$F'(P) = 1 + r - \frac{2rP}{K}.$$

An equilibrium is stable if $|F'(P^*)| < 1$ and unstable if $|F'(P^*)| > 1$. At the two equilibria:

$$F'(0) = 1 + r = 2.4, \qquad |2.4| > 1 : \text{ unstable.}$$

$$F'(K) = 1 + r - 2r = 1 - r = -0.4, \qquad |-0.4| = 0.4 < 1 : \text{ stable.}$$

The negative sign of $F'(K)$ does not matter; only its size does.

Problem 2 — 11 points

$\dfrac{dP}{dt} = f(P) = rP\Big(\dfrac{P}{A} - 1\Big)\Big(1 - \dfrac{P}{K}\Big)$, $r = 1$, $A = 3$, $K = 10$.

(a) **[4 pts]**

Equilibria are the values of P where $dP/dt = 0$. The right-hand side is a product, so one factor must vanish:

$$P = 0, \qquad \frac{P}{3} - 1 = 0 \;\Rightarrow\; P = 3, \qquad 1 - \frac{P}{10} = 0 \;\Rightarrow\; P = 10.$$

The equilibria are $P = 0$, $P = 3$, and $P = 10$.

(b) **[5 pts]**

The sign of f on each interval between equilibria gives the direction in which P moves:

interval	P	$\frac{P}{3} - 1$	$1 - \frac{P}{10}$	sign of f	P moves
$(0, 3)$	$+$	$-$	$+$	$-$	down
$(3, 10)$	$+$	$+$	$+$	$+$	up
$(10, \infty)$	$+$	$+$	$-$	$-$	down

On the phase line, the arrows point toward $P = 0$ (from the right) and toward $P = 10$ (from both sides), and away from $P = 3$ on both sides. Therefore $P = 0$ is **stable**, $P = 3$ is **unstable**, and $P = 10$ is **stable**.

(c) **[2 pts]**

A population starting below $P = 3$ is in the interval $(0, 3)$, where $f < 0$, so it decreases toward 0 and goes extinct. The value $P = 3$ is the minimum viable population size.

Problem 3 — 10 points

20% *of juveniles survive to become adults;* 50% *of adults survive and remain adults; each adult produces* 2.5 *juveniles per year.* $n_{t+1} = Ln_t$ *with* $n_t = \binom{J_t}{A_t}$.

(a) **[4 pts]**

New juveniles come only from adults, and no juvenile stays a juvenile:

$$J_{t+1} = 0 \cdot J_t + 2.5\, A_t.$$

Next year's adults are the surviving juveniles plus the surviving adults:

$$A_{t+1} = 0.2\, J_t + 0.5\, A_t.$$

Reading off the coefficients row by row gives

$$L = \begin{pmatrix} 0 & 2.5 \\ 0.2 & 0.5 \end{pmatrix}.$$

(b) **[5 pts]**

For a 2×2 matrix the eigenvalues satisfy $\lambda^2 - T\lambda + D = 0$, where $T = \operatorname{tr} L$ and $D = \det L$. Here $T = 0 + 0.5 = 0.5$ and $D = (0)(0.5) - (2.5)(0.2) = -0.5$, so

$$\lambda^2 - 0.5\lambda - 0.5 = 0.$$

By the quadratic formula,

$$\lambda = \frac{0.5 \pm \sqrt{0.25 + 2}}{2} = \frac{0.5 \pm 1.5}{2},$$

so $\lambda = 1$ and $\lambda = -0.5$. (Check: $(\lambda - 1)(\lambda + 0.5) = \lambda^2 - 0.5\lambda - 0.5$.)

(c) **[1 pt]**

The long-run behavior is set by the dominant (largest) eigenvalue, $\lambda_1 = 1$. A dominant eigenvalue above 1 means growth, below 1 means decline, and equal to 1 means the total population neither grows nor shrinks. This population is **exactly stable** in the long run.

Problem 4 — 9 points

$\dfrac{dR}{dt} = -R + 4J$, $\dfrac{dJ}{dt} = -2R - 3J$.

(a) **[4 pts]**

The coefficient matrix is

$$A = \begin{pmatrix} -1 & 4 \\ -2 & -3 \end{pmatrix}.$$

Then

$$T = \operatorname{tr} A = -1 + (-3) = -4, \qquad D = \det A = (-1)(-3) - (4)(-2) = 3 + 8 = 11.$$

(b) **[5 pts]**

Apply the tests in order.

1. Sign of D: $D = 11 > 0$, so the origin is not a saddle.
2. Sign of T: $T = -4 < 0$, so it is stable.
3. Discriminant: $T^2 - 4D = 16 - 44 = -28 < 0$, so the eigenvalues are complex and the origin is a spiral.

The origin is a **stable spiral**. (Check: $\lambda = \dfrac{-4 \pm \sqrt{-28}}{2} = -2 \pm i\sqrt{7}$, with negative real part and nonzero imaginary part.)

Problem 5 — 10 points

$\dfrac{dx}{dt} = x(\alpha - \beta y)$, $\dfrac{dy}{dt} = y(\delta x - \gamma)$, $\alpha = 3$, $\beta = 1$, $\gamma = 2$, $\delta = 0.5$.

(a) **[3 pts]**

At a coexistence equilibrium both x and y are nonzero, so both bracketed factors must vanish:

$$\alpha - \beta y = 0 \;\Rightarrow\; y^* = \frac{\alpha}{\beta} = 3, \qquad \delta x - \gamma = 0 \;\Rightarrow\; x^* = \frac{\gamma}{\delta} = 4.$$

So $(x^*, y^*) = (4, 3)$.

(b) **[5 pts]**

Write $f = \alpha x - \beta xy$ and $g = \delta xy - \gamma y$. The Jacobian is the matrix of partial derivatives, where each partial derivative holds the other variable fixed:

$$J = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix} = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix}.$$

At $(4, 3)$: $\alpha - \beta y = 3 - 3 = 0$, $-\beta x = -4$, $\delta y = 1.5$, and $\delta x - \gamma = 2 - 2 = 0$. So

$$J = \begin{pmatrix} 0 & -4 \\ 1.5 & 0 \end{pmatrix}, \qquad T = 0, \qquad D = (0)(0) - (-4)(1.5) = 6.$$

(c) **[2 pts]**

$T = 0$ and $D = 6 > 0$, so the equilibrium is a **center**.