MATH 448 — Midterm 1 — Solutions

Problem 1 — 10 points

$P_{t+1} = P_t\big(1 + r(1 - P_t/K)\big)$, $r = 2.5$, $K = 20$.

(a) **[3 pts]**

Solution. $F(P) = P$ gives $rP(1 - P/K) = 0$, so $P^* = 0$ and $P^* = K = 20$.

(b) **[7 pts]**

Solution. $F'(P) = 1 + r - 2rP/K$. $F'(0) = 3.5$ and $|3.5| > 1$: unstable. $F'(K) = 1 - r = -1.5$ and $|-1.5| > 1$: unstable.

Problem 2 — 11 points

$\dfrac{dP}{dt} = rP\Big(\dfrac{P}{A} - 1\Big)\Big(1 - \dfrac{P}{K}\Big)$, $r = 1$, $A = 2$, $K = 12$.

(a) **[4 pts]**

Solution. Set $dP/dt = 0$. The factors vanish at $P = 0$, $P = 2$, $P = 12$.

(b) **[5 pts]**

Solution. On $(0, 2)$: $(+)(-)(+)$ = negative. On $(2, 12)$: $(+)(+)(+)$ = positive. On $(12, \infty)$: $(+)(+)(-)$ = negative. So $P = 0$ and $P = 12$ are stable; $P = 2$ is unstable.

(c) **[2 pts]**

Solution. A population starting below $P = 2$ declines to extinction: 2 is the minimum viable population size.

Problem 3 — 10 points

Juveniles: 20% *survive to become adults. Adults:* 30% *survive and remain adults; each produces* 4.4 *juveniles on average.* $n_{t+1} = Ln_t$, $n_t = \binom{J_t}{A_t}$.

(a) **[4 pts]**

Solution. $L = \begin{pmatrix} 0 & 4.4 \\ 0.2 & 0.3 \end{pmatrix}$.

(b) **[5 pts]**

Solution. $T = 0.3$, $D = -0.88$, so $\lambda^2 - 0.3\lambda - 0.88 = 0$ and $\lambda = \dfrac{0.3 \pm \sqrt{0.09 + 3.52}}{2} = \dfrac{0.3 \pm 1.9}{2}$, giving $\lambda = 1.1$ and $\lambda = -0.8$.

(c) **[1 pt]**

Solution. The dominant eigenvalue is $1.1 > 1$, so the population is growing (about 10% per year in the long run).

Problem 4 **9 points**

$\frac{dR}{dt} = R + 3J$, $\frac{dJ}{dt} = 2R + 2J$.

(a) **[4 pts]**

Solution. $A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$, so $T = 1 + 2 = 3$ and $D = (1)(2) - (3)(2) = -4$.

(b) **[5 pts]**

Solution. $D = -4 < 0$: a saddle, always unstable regardless of T. (Directly: $\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$, so $\lambda = 4, -1$, real with opposite signs.)

Problem 5 **10 points**

$\frac{dx}{dt} = x(\alpha - \beta y)$, $\frac{dy}{dt} = y(\delta x - \gamma)$, $\alpha = 4$, $\beta = 2$, $\gamma = 3$, $\delta = 1$.

(a) **[3 pts]**

Solution. $(x^*, y^*) = (\gamma/\delta,\ \alpha/\beta) = (3/1,\ 4/2) = (3, 2)$.

(b) **[5 pts]**

Solution. $J = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix}$, so at $(3, 2)$, $J = \begin{pmatrix} 0 & -6 \\ 2 & 0 \end{pmatrix}$ with $T = 0$ and $D = 0 - (-6)(2) = 12$.

(c) **[2 pts]**

Solution. $T = 0$ and $D = 12 > 0$: a center.