MATH 448 — Midterm 1 Study Guide

0. One method for every model

Finding equilibria and testing stability

Every stability question in this course follows the same four steps:

(i) Find all equilibria.
(ii) Linearize near each equilibrium (find the Jacobian there).
(iii) Find the eigenvalues of the linearization.
(iv) Decide stability from the eigenvalues.

The steps are the same for every model. Only step (iv) changes, depending on whether time is continuous or discrete.

- **Continuous time:** stable if all eigenvalues have negative real part, unstable if any has positive real part.
- **Discrete time:** stable if all eigenvalues have absolute value less than 1, unstable if any has absolute value greater than 1.

Model	Equilibria	Linearization	Eigenvalue(s)	Stable if	Unstable if
Discrete, one variable: $P_{t+1} = F(P_t)$	Solve $F(P) = P$	$p_{t+1} \approx F'(P^*)\, p_t$	$F'(P^*)$	$\lvert F'(P^*)\rvert < 1$	$\lvert F'(P^*)\rvert > 1$
Continuous, one variable: $dP/dt = f(P)$	Solve $f(P) = 0$	$dp/dt \approx f'(P^*)\, p$	$f'(P^*)$	$f'(P^*) < 0$	$f'(P^*) > 0$
Discrete, linear: $n_{t+1} = Ln_t$	$n = 0$	Already linear	Eigenvalues of L ($\lambda^2 - T\lambda + D = 0$)	Every $\lvert\lambda\rvert < 1$ (the population dies out)	Some $\lvert\lambda\rvert > 1$ (the population grows)
Continuous, two variables: $dx/dt = f(x,y)$, $dy/dt = g(x,y)$	Solve $f = 0$ and $g = 0$ together	Jacobian J at the equilibrium	Eigenvalues of J ($\lambda^2 - T\lambda + D = 0$)	Every eigenvalue has negative real part. For 2×2: $T < 0$ and $D > 0$	Some eigenvalue has positive real part. For 2×2: $D < 0$, or $D > 0$ and $T > 0$

- The boundary cases are not decided by the linear test: $\lvert F'\rvert = 1$, $f' = 0$, an eigenvalue with $\lvert\lambda\rvert = 1$, or an eigenvalue with real part 0. In the plane, the last case means $D = 0$, or $T = 0$ with $D > 0$.
- For the one-variable continuous model, the phase line (the sign of f on each side of the equilibrium) settles the boundary case.
- The largest $\lvert\lambda\rvert$ of a projection matrix is its dominant eigenvalue, and it gives the long-run growth rate.

Can you... find the equilibria of a model of each type and decide their stability?

1. Discrete growth models

Lectures 1–2

Models

Malthusian: $P_{t+1} = rP_t$ (exponential: $r > 1$ grows, $r < 1$ decays, $r = 1$ constant).
Logistic: $P_{t+1} = P_t\big(1 + r(1 - P_t/K)\big)$. Equilibria: $P^* = 0$, $P^* = K$.

Can you... set up a discrete model from a verbal description of growth and limitation?

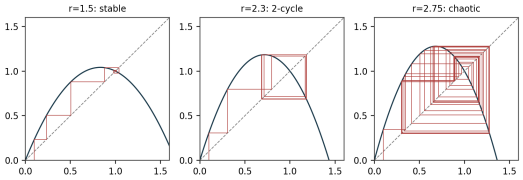
2. Equilibrium, stability, and chaos (discrete)

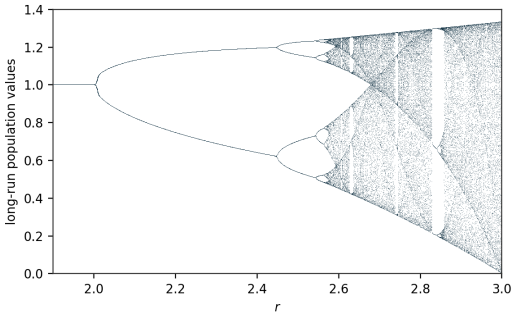
Lecture 3

Stability test

P^* is an equilibrium if $F(P^*) = P^*$. It is **stable** if $\lvert F'(P^*)\rvert < 1$, **unstable** if $\lvert F'(P^*)\rvert > 1$; the test is **inconclusive** if $\lvert F'(P^*)\rvert = 1$ (exactly where bifurcations happen). For the logistic map, $F'(K) = 1 - r$: stable for $0 < r < 2$.

Past $r = 2$: K loses stability, a stable 2-cycle appears (period doubling), then 4, 8, ..., chaos onset near $r \approx 2.570$. **Chaos** means deterministic but highly sensitive to initial conditions, not randomness.





Can you... apply both the noncalculus (substitute $P^* + p$ and keep the linear term) and calculus stability tests? read a bifurcation diagram? explain why chaos is not the same as randomness? (HW2)

3. Continuous models and the Ricker fix

Lecture 4

Continuous logistic

$$\frac{dP}{dt} = rP\Big(1 - \frac{P}{K}\Big), \qquad \text{equilibria } P^* = 0, K.$$

Stability test: P^* stable if $f'(P^*) < 0$, unstable if $f'(P^*) > 0$. Here $f'(K) = -r < 0$ *for every* $r > 0$: unlike the discrete model, K is always stable: no bifurcation, no chaos.

Dimension and chaos: a 1D continuous system cannot oscillate; a 2D system can oscillate but not be chaotic (Poincaré–Bendixson); chaos needs 3 or more coupled continuous variables (Lorenz system). **Can you...** do a phase-line analysis and classify stability without a formula? explain why continuous and discrete logistic models behave differently as r grows? (HW3)

4. Euler's method and evolutionary game theory

Lecture 5

Euler's method

$P_{t+1} = P_t + h \cdot f(P_t)$ turns an ODE into a discrete model with step size h. Setting $h = 1$ recovers the original discrete model exactly.

Replicator equation

Strategy, frequency x*, fitness* $f(x)$*, payoff*: terms for variation within a population. For two strategies with fitnesses $f_A(x), f_B(x)$,

$$\frac{dx}{dt} = x(1-x)\big(f_A(x) - f_B(x)\big).$$

$x = 0, 1$ are always equilibria; interior equilibria solve $f_A = f_B$.

Hawk–Dove: resource V, injury cost $C > V$; stable interior mix $x^* = V/C$. Stag Hunt (HW4) has the opposite structure: both pure strategies stable, interior point unstable (bistability). **Can you...** build a payoff matrix from a verbal description and derive the resulting replicator equation? distinguish a single stable interior mix from bistability? (HW3, HW4)

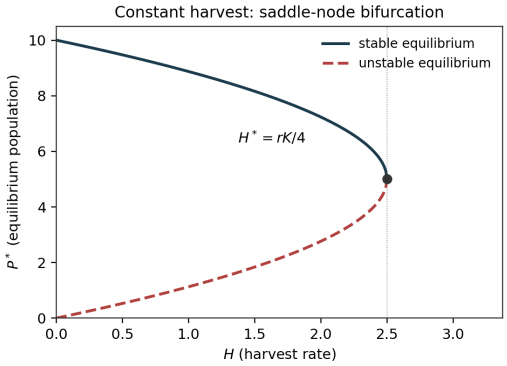
5. Harvesting and bifurcation

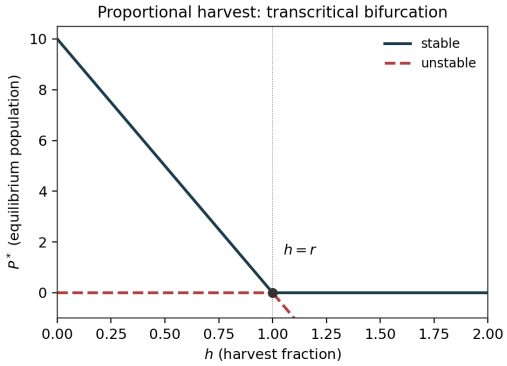
Lecture 6

Two harvesting policies

Constant: $\frac{dP}{dt} = rP(1 - P/K) - H$. Two equilibria merge and vanish at $H^* = rK/4$ (**saddle-node bifurcation**): cross it and the population crashes with no warning.

Proportional: $\frac{dP}{dt} = rP(1-P/K)-hP$. Equilibria 0 and $K(1-h/r)$ swap stability at $h^* = r$ (**transcritical bifurcation**): decline is gradual, no hidden threshold.





Can you... find the maximum sustainable harvest for each policy, and explain in words why the two failure modes differ? (HW4)

6. Structured populations and eigenvalues

Lecture 7

Projection matrices

Stage vector n_t, projection matrix L: $n_{t+1} = Ln_t$. The **dominant eigenvalue** λ_1 is the long-term growth rate; its eigenvector is the **stable stage distribution**. For a 2×2 matrix, find λ via the characteristic equation $\lambda^2 - T\lambda + D = 0$ (quadratic formula).

Why it works: writing n_0 in terms of L's eigenvectors and applying L repeatedly, the dominant term eventually swamps the others. This is why λ_1 and its eigenvector describe the long run regardless of the starting distribution. **Can you...** build a 2×2 projection matrix from a verbal description of survival and reproduction, and find both eigenvalues by hand?

7. Linear systems in the plane

Lecture 8

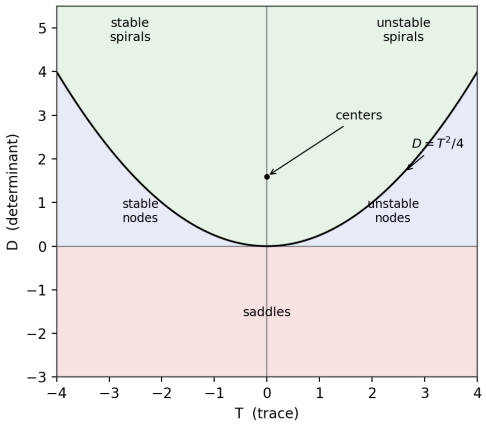
Classification by trace and determinant

For $\dot{\mathbf{x}} = A\mathbf{x}$, with $T = \operatorname{tr} A$, $D = \det A$: eigenvalues satisfy $\lambda^2 - T\lambda + D = 0$, so they sum to T and multiply to D (Vieta's formulas). Classify in order:

(a) $D < 0$: a **saddle**, always unstable. $D > 0$: continue.

(b) $T < 0$: stable. $T > 0$: unstable. $T = 0$: a **center** (purely imaginary eigenvalues, perpetual oscillation).

(c) (if $T \neq 0$) $T^2 - 4D \geq 0$: a **node** (real eigenvalues). $T^2 - 4D < 0$: a **spiral** (complex eigenvalues).



For a symmetric matrix $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, the eigenvalues are $a+b$ and $a-b$ by inspection (eigenvectors $(1,1)$, $(1,-1)$), so no characteristic equation is needed. **Can you…** classify a 2×2 linear system's equilibrium from T and D alone, without finding the eigenvalues explicitly? (HW5)

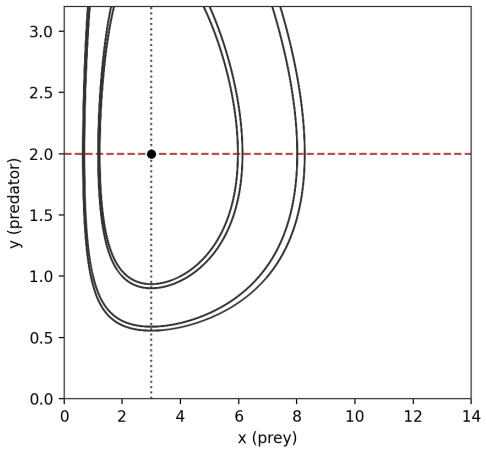
8. Lotka–Volterra predator–prey

Lecture 9

The model

$$\frac{dx}{dt} = x(\alpha - \beta y), \qquad \frac{dy}{dt} = y(\delta x - \gamma).$$

Equilibria: $(0,0)$ and $\big(\gamma/\delta,\ \alpha/\beta\big)$. Jacobian at coexistence always has $T = 0$, $D = \alpha\gamma > 0$: a **center**, for *every* choice of positive parameters.



Constant-yield harvesting of either species destabilizes this center (makes $T > 0$); proportional harvesting does not (HW5, Lecture 9 discussion). **Can you…** find both equilibria and the Jacobian at coexistence by hand, and classify the coexistence equilibrium from T and D?